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Spatially Varying Coefficients Models: A Comparison of Maximum Likelihood Estimators with other Estimators

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1. Introduction

We consider the multiple regression model for spatial data. Often, the relationship between the explanatory variables and the dependent variable is not constant over space. Our goal is to model this form of non-stationarity using spatially varying coefficients (SVC) models which generalize the classical linear regression model. Thus, the underlying model formula is

$$y(s) = \beta_1(s)x^{(1)}(s) + \beta_2(s)x^{(2)}(s) + \cdots + \beta_p(s)x^{(p)}(s) + \epsilon(s).$$

where s is a location in a domain $D \subseteq R^2$. This allows for making more accurate predictions and for discovering relationships that vary over space. For the coefficients and the error term we assume Gaussian random fields (GRF) with exponential correlation function $\phi_\rho(d) = \exp(-d/\rho)$, where d is the distance between two observations. Hence we parametrize $\beta_j(\cdot) \sim N(\mu_j, \sigma_j^2 \phi_{\rho_j}(\cdot))$.

Effect (j)	Range ρ_j	Variance σ_j^2	Mean μ_j
1	0.10	0.20	0.00
2	0.20	0.10	0.00
3	0.15	0.05	0.00
nugget	---	0.03	---

Tab.1: Example of a SVC Model parametrization with $p = 3$. We will use this model to compare methods.

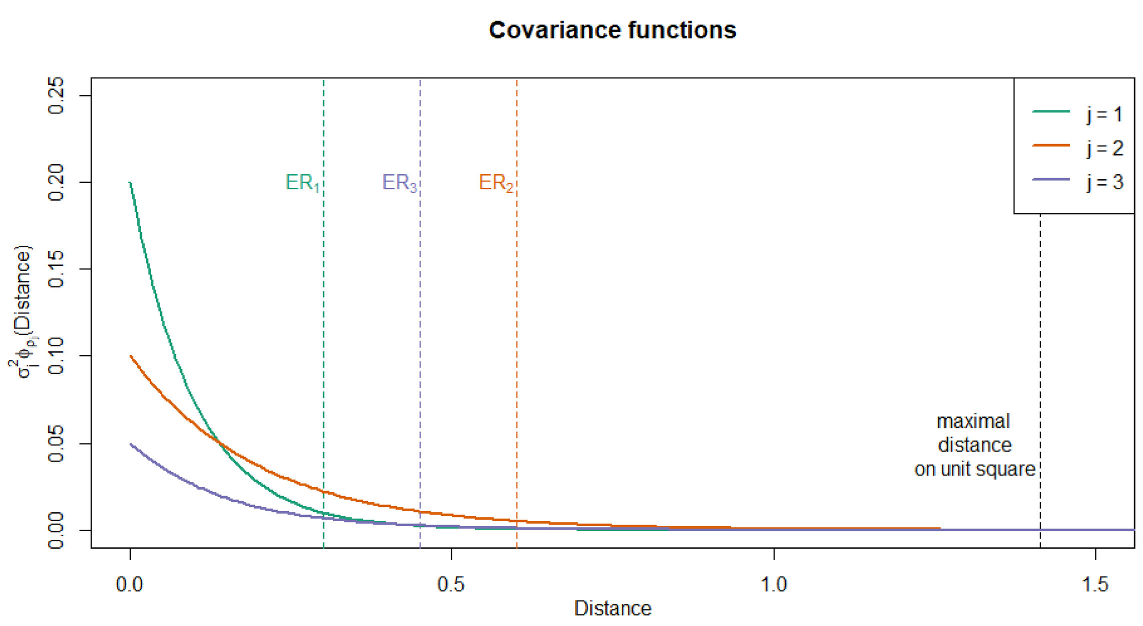

Fig.1: Visualization of covariance functions parametrized in **Tab.1**. The effective ranges of the covariance functions are marked with ER. The domain D we work with is the unit square, hence the maximal distance of $\sqrt{2}$.

2. Our Model and Approach

With

- the covariate matrix $\boldsymbol{X} := (\text{diag}\,\boldsymbol{X}^{(1)}(s)|\ldots|\text{diag}\,\boldsymbol{X}^{(p)}(s))$
- the GRFs covariance matrices $\boldsymbol{\Sigma}_j$ depending on ρ_j and σ_j^2
- the mean vector $\boldsymbol{\mu} := (\mu_1 \mathbf{1}, \ldots, \mu_p \mathbf{1})$
- the parameter set $\boldsymbol{\theta}$ consisting of all $\rho_j, \sigma_j^2, \mu_j$ and σ_{nugget}^2

and the model formula from above we get

$$\boldsymbol{y} \mid \boldsymbol{X}, \boldsymbol{\theta} \sim N\left(\boldsymbol{X\mu}, \sum_{j=1}^{p} \boldsymbol{X}^{(j)} \boldsymbol{\Sigma}_j \boldsymbol{X}^{(j)\prime} + \boldsymbol{\Sigma}_{nugget}\right).$$

We can define a likelihood function and using a general purpose quasi Newton method we obtain a maximum likelihood estimator (MLE) for θ. Once the parameters $\hat{\theta}_{MLE}$ have been estimated, one can use the properties of a joint normal distribution to estimate $\beta_j(s)|\,y$ like one would with regular kriging in geo-statistics.

3. Existing Methods

There exists several methods and techniques on how to model and predict SVC. We shortly list the most common ones.

Geographically Weighted Regression (GWR)

- It does not assume any kind of distribution for the marginal effects. Rather, it models SVC by a weighted regression.
- Should be used for explorative analysis only.
- Available in R (package `spgwr`) and arcGIS.

Integrated Nested Laplacian Approximation (INLA)

- Bayesian method that can approximate our model with a Markov Field via an SPDE.
- Available in R (`www.r-inla.org`)

Eigenvector Spatial Filtering (ESF)

- Regression method that has been extended to work with random effects and thus can model SVCs.
- Available in R (package `spmoran`)

4. Simulation Study

To compare the methods, we perform a simulation study on synthetic data. We work on a unit square using the model parameterized in **Tab.1**. The procedure is as follows:

1. Covariate $\boldsymbol{X}^{(1)}$ is set to 1 in all simulations to generate an intercept, the covariates $\boldsymbol{X}^{(2)}$ and $\boldsymbol{X}^{(3)}$ are sampled from a standard normal distribution. With the covariates and effects, we calculate the response $\boldsymbol{y}$.
2. The data is then partitioned into a training ($n_{train} = 1250$) and test ($n_{test} = 625$) set.
3. If the method allows for estimating the covariance parameters based on the training set, results are shown.
4. The out-of-sample MSE is calculated on the test set both for the estimated effects and the response.

This procedure is repeated $N = 100$ times. Results for methods MLE, INLA and ESF are shown in **Fig.2** and **Fig.3**.

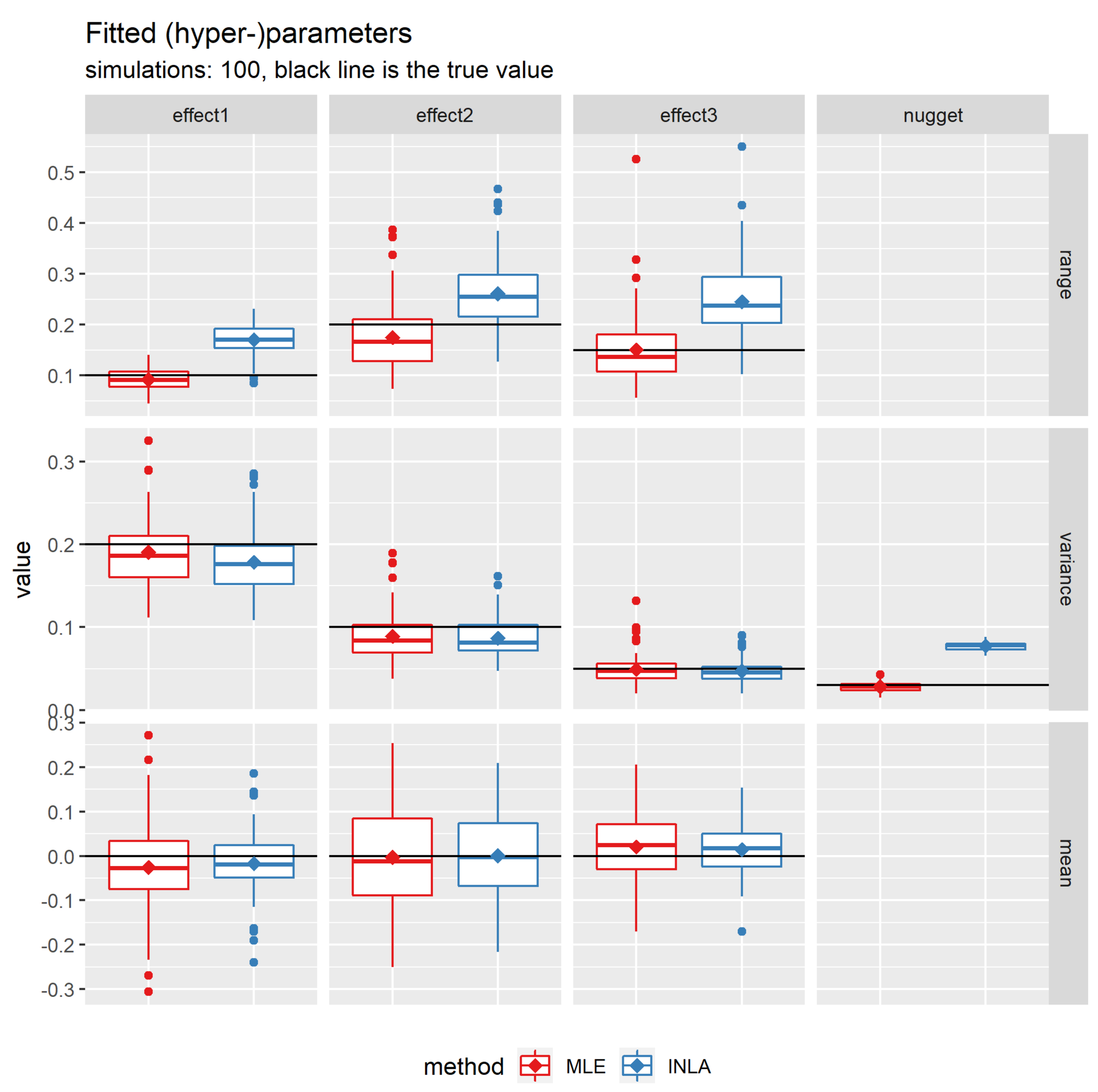

Fig.2: Simulation estimates for $\mu_j, \sigma_j^2, \rho_j$ for $j = 1, 2, 3$ as well as the nugget. The black horizontal lines are the true values, cf. **Tab.1** for true values. The ESF method does not provide estimates for the parameters.

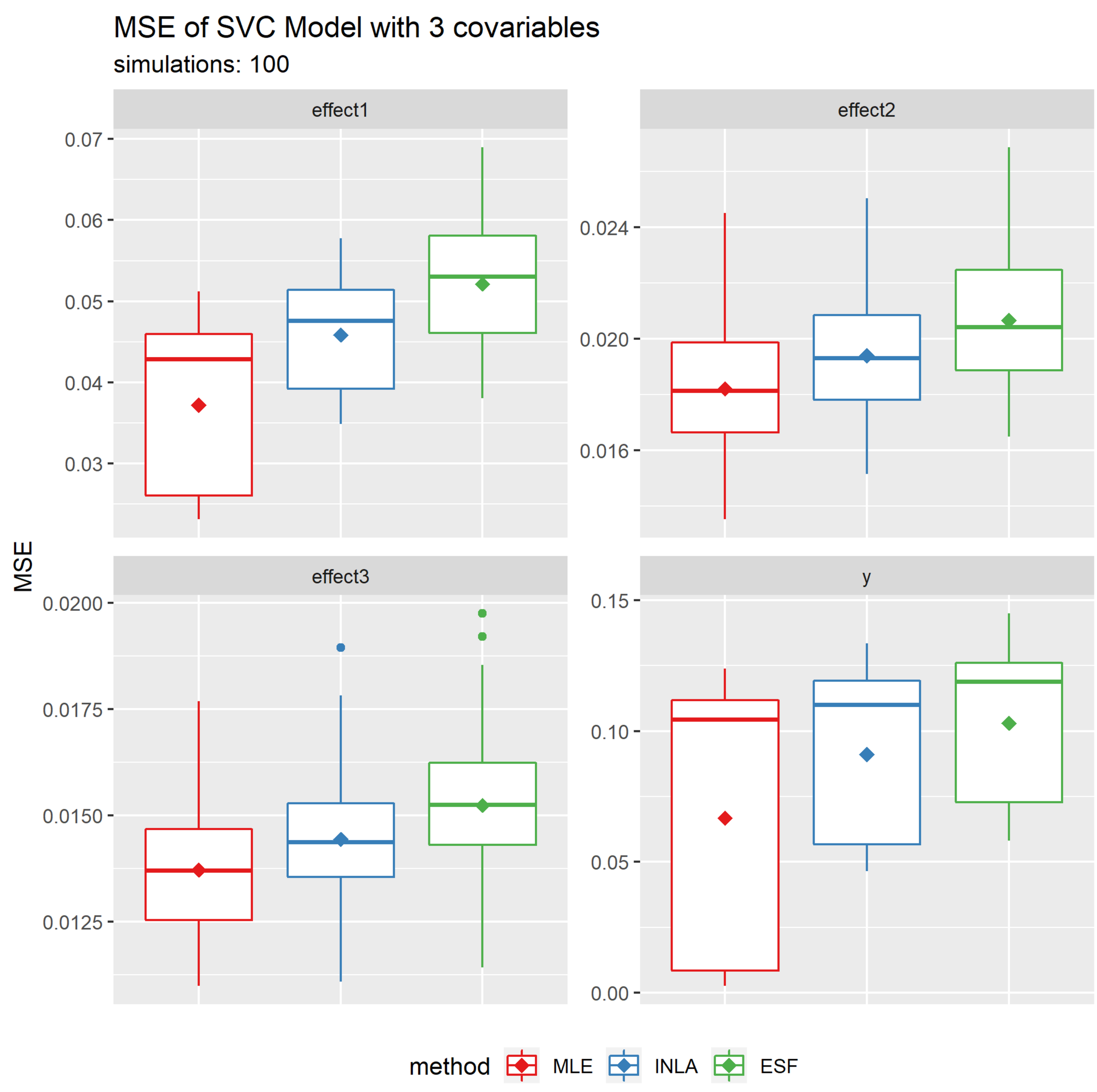

Fig.3: Out-of-sample MSE for SVC and response for parameters given in **Tab.1**.

Conclusion and Future Work

First results are promising in parameter and effect estimation. Methodology for large data sets as well as simulations with more SVC are needed. In **Fig.4** we show the out-of-sample MSE for a similar model with $p = 10$ explanatory variables. Finally, performance on real data sets has to be evaluated.

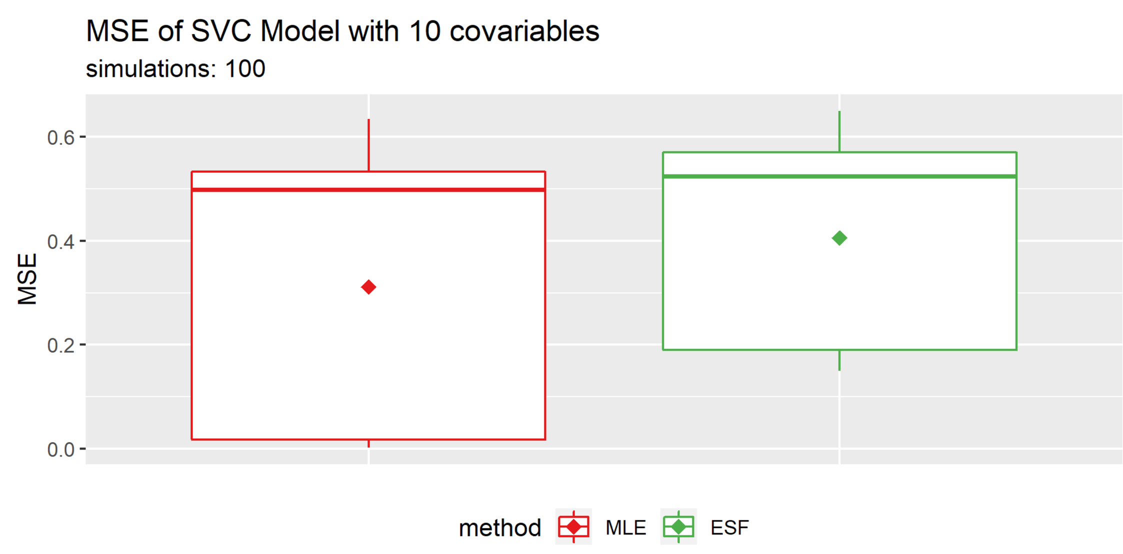

Fig.4: Out-of-sample MSE for response for model with $p = 10$ SVC, cf. Section 2. INLA could not finish.

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